

6.1.1

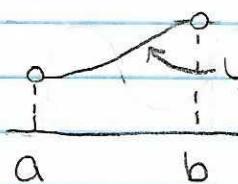
Let f and g be bounded functions on $[a, b]$.

Prove $\|f+g\|_U \leq \|f\|_U + \|g\|_U$

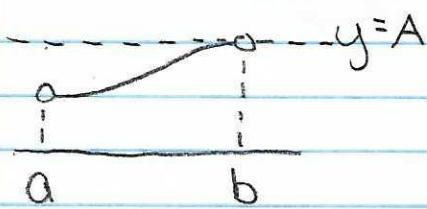
Recall:

\sup = least upper bound. There is no portion of the graph that lies above the \sup (supremum).

Ex 1: Understanding Supremum



Find a line on the graph where if you go higher, there will be no portion of the original graph $f(x)$



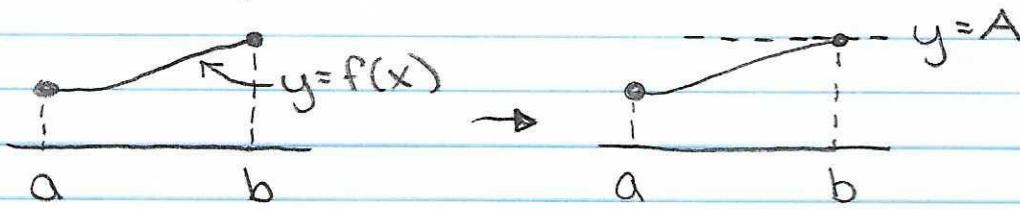
$y = A$ is the only possible line to draw where the original function will not be above the line.

Therefore, $A = \text{supremum value of } f(x) \text{ on the interval } (a, b)$

- However, A is not the maximum because it is on an open interval where b is not included (represented by an open circle on the graph or parentheses (a, b))

Let's look at 2 examples where the supremum is also the maximum

Ex 2: Supremum vs. Maximum:



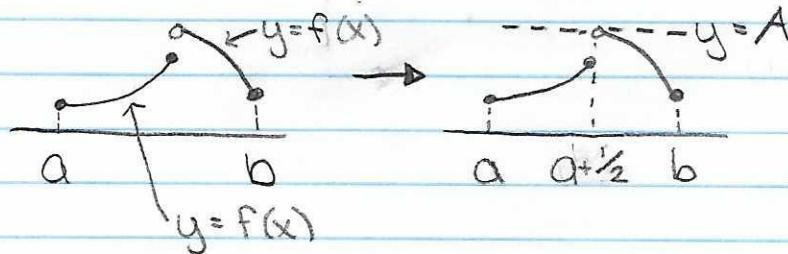
This is an obvious example, as it mimics the first one, except this is plotted on closed intervals (represented by filled-in circles on the graph or brackets $[a, b]$).

Since none of the graph lies above $y = A$...

A = supremum value of $f(x)$ on the interval $[a, b]$

A = Maximum since b is included

Ex 3: Sup vs. Max Again:



- Although this looks different, you apply the same technique.

A = supremum value of $f(x)$ on the interval $(a + \frac{1}{2}, b]$, but not the maximum.

"Proof":

$$\|f+g\|_U \leq \|f\|_U + \|g\|_U$$

$$= \sup_{x \in [a,b]} \{ |f(x) + g(x)| \} \leq \sup_{x \in [a,b]} \{ |f(x)| \} + \sup_{x \in [a,b]} \{ |g(x)| \}$$

recall since the sup is on a closed interval, it is also the max

If sup is also the max, then:

$$|f(x)| \leq \sup_{x \in [a,b]} \{ |f(x)| \} \text{ and } |g(x)| \leq \sup_{x \in [a,b]} \{ |g(x)| \}$$

Therefore,

$$|f(x)| + |g(x)| \leq \sup_{x \in [a,b]} \{ |f(x)| \} + \sup_{x \in [a,b]} \{ |g(x)| \}$$

Thus,

$|f(x)| + |g(x)|$ is bounded from above by
 $\sup_{x \in [a,b]} \{ |f(x)| \} + \sup_{x \in [a,b]} \{ |g(x)| \}$

So,

$$\sup_{x \in [a,b]} \{ |f(x) + g(x)| \} \leq \sup_{x \in [a,b]} \{ |f(x)| \} + \sup_{x \in [a,b]} \{ |g(x)| \}$$

6.1.2

a.) Find the pointwise limit $e^{x/n}/n$ for $x \in \mathbb{R}$

* To find a pointwise limit means to "fix x and let n go to infinity"

① When $x < 0$:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{e^{x/n}}{n} &= \left(\lim_{n \rightarrow \infty} e^{-1/100} \right) \left(\lim_{n \rightarrow \infty} \frac{1}{100} \right) \\ &= (1)(0) \\ &= 0\end{aligned}$$

② When $x = 0$:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{e^{x/n}}{n} &= \left(\lim_{n \rightarrow \infty} e^0 \right) \left(\lim_{n \rightarrow \infty} \frac{1}{100} \right) \\ &= (1)(0) \\ &= 0\end{aligned}$$

③ When $x > 0$:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{e^{x/n}}{n} &= \left(\lim_{n \rightarrow \infty} e^{1/100} \right) \left(\lim_{n \rightarrow \infty} \frac{1}{100} \right) \\ &= (1)(0) \\ &= 0\end{aligned}$$

For all $x \in \mathbb{R}$ the limit converges to 0.
 $f(x) = 0$

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b.) Is the limit uniform on \mathbb{R} ?

Recall:

A sequence of bounded functions $f_n: S \rightarrow \mathbb{R}$ converges uniformly to $f: S \rightarrow \mathbb{R}$ if and only if $\lim_{n \rightarrow \infty} \|f_n - f\|_u = 0$

① Must find min and max:

$$\begin{aligned}|f_n(x) - f(x)| \\= |e^{x/n}/n - 0| = |e^{x/n}/n| = e^{x/n}/n\end{aligned}$$

$$f_n(x) = e^{x/n}/n$$

$$f_n'(x) = e^{x/n}/n^2$$

$e^{x/n}/n^2 = 0 \leftarrow$ set equal to 0 to find critical numbers

$$e^{x/n}/n^2 = 0$$

$$e^{x/n} = 0$$

$$\sqrt[n]{e^x} = 0$$

$$e^x = 0^n$$

$$e^x = 0$$

$$\ln(e^x) = \ln(0)$$

$$x = \text{undefined}$$

Therefore, $e^{x/n}/n$ is not uniform on \mathbb{R}

6.1.2

c.) Is the limit uniform on $[0,1]$?

Recall:

The sequence $\{f_n\}$ converges uniformly to f if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in S$

Goal:

$$|e^{x/n}/n - 0| < \epsilon$$

$$|e^{x/n}/n| < \epsilon$$

$$e^{x/n}/n \leq e^x/n < \epsilon$$

$$e^x/n \leq e^x/n < \epsilon$$

$$e^x < \epsilon \cdot n$$

$$e^x/\epsilon < n, N$$

Let $\epsilon > 0$. Choose $N > e/\epsilon$

Then $\forall n > N \quad \forall x \in [0,1]$

$$|e^{x/n}/n - 0| = |e^{x/n}/n| = e^{x/n}/n$$

$$e^{x/n}/n \leq e^x/n \leq e^x/\epsilon = \frac{e^x}{\epsilon} \rightarrow \text{show this is } < \epsilon$$

Since $n > N > e/\epsilon$, so

$$n > e/\epsilon$$

$$n \cdot \epsilon > e$$

$$\epsilon > e/n$$

$$\begin{aligned} \text{Hence, } |e^{x/n}/n - 0| \\ = e^{x/n}/n \leq e^x/n < \epsilon \end{aligned}$$